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ON THE HYPOTHESES WHICH LIE AT THE FOUNDATIONS OF GEOMETRY

(Translated from the German by Professor Henry S. White, Vassar College, Poughkeepsie, N. Y.)

The paper here translated is Riemann's *Probe-Vorlesung*, or formal initial lecture on becoming Privat-Docent. It is extraordinary in scope and originality and it paved the way for the now current theories of hyperspace and relativity. It was read on the 10th of June, 1854, for the purpose of Riemann's "Habilitation" with the philosophical faculty of Gottingen. This explains the form of presentation, in which analytic investigations could be only indicated; some elaborations of them are to be found in the "*Commentatio matbematica, qua respondere tentatur quaestioni ab Illma Academia Parisiensi propositae*" etc., and in the appendix to that paper. It appears in vol. XIII of the *Abbandlungen* of the Royal Society of Sciences of Gottingen.

Plan of the Investigation

It is well known that geometry presupposes not only the concept of space but also the first fundamental notions for constructions in space as given in advance. It gives only nominal definitions for them, while the essential means of determining them appear in the form of axioms. The relation of these presuppositions is left in the dark; one sees neither whether and in how far their connection is necessary, nor a priori whether it is possible.

From Euclid to Legendre, to name the most renowned of modern writers on geometry, this darkness has been lifted neither by the mathematicians nor by the philosophers who have labored upon it. The reason of this lay perhaps in the fact that the general concept of multiply extended magnitudes, in which spatial magnitudes are comprehended, has not been elaborated at all. Accordingly I have proposed to myself at first the problem of constructing the concept of a multiply extended magnitude out of general notions of quantity. From this it will result that a multiply extended magnitude is susceptible of various metric relations and that space accordingly constitutes only a particular case of a triply extended magnitude. A necessary sequel of this is that the propositions of geometry are not derivable from general concepts of quantity, but that those properties by which space is distinguished from other conceivable triply extended magnitudes can be gathered only from experience. There arises from this the problem of searching out the simplest facts by which the metric relations of space can be determined, a problem which in nature of things is not quite definite; for several systems of simple facts can be stated which would suffice for determining the metric relations of space; the most important for present purposes is that laid down for foundations by Euclid. These facts are, like all facts, not necessary but of a merely empirical certainty; they are hypotheses; one may therefore inquire into their probability, which is truly very great within the bounds of observation, and thereafter decide concerning the admissibility of protracting them outside the limits of observation, not only toward the immeasurably large, but also toward the immeasurably small.

I. The Concept of n-fold Extended Manifold

While I now attempt in the first place to solve the first of these problems, the development of the concept of manifolds multiply extended, I think myself the more entitled to ask considerate judgment inasmuch as I have had little practice in such matters of a philosophical nature, where the difficulty lies more in the concepts than in the construction, and because I have not been able to make use of any preliminary studies whatever aside from some very brief hints which Privy Councilor Gauss has given on the subject in his second essay on biquadratic residues and in his Jubilee booklet, and some philosophical investigations of Herbart.

1

Notions of quantity are possible only where there exists already a general concept which allows various modes of determination. According as there is or is not found among these modes of determination a continuous transition from one to another, they form a continuous or a discrete manifold; the individual modes are called in the first case points, in the latter case elements of the manifold. Concepts whose modes of determination form a discrete manifold are so numerous, that for things arbitrarily given there can always be found a concept, at least in the more highly developed languages, under which they are comprehended (and mathematicians have been able therefore in the doctrine of discrete quantities to set out without scruple from the postulate that given things are to be considered as all of one kind); on the other hand there are in common life only such infrequent occasions to form concepts whose modes of determination form a continuous manifold, that the positions of objects of sense, and the colors, are probably the only simple notions whose modes of determination form a multiply extended manifold. More frequent occasion for the birth and development of these notions is first found in higher mathematics.

Determinate parts of a manifold, distinguished by a mark or by a boundary, are called quanta. Their comparison as to quantity comes in discrete magnitudes by counting, in continuous magnitude by measurement. Measuring consists in superposition of the magnitudes to be compared; for measurement there is requisite some means of carrying forward one magnitude as a measure for the other. In default of this, one can compare two magnitudes only when the one is a part of the other, and even then one can only decide upon the question of more and less, not upon the question of how many. The investigations which can be set on foot about them in this case form a general part of the doctrine of quantity independent of metric determinations, where magnitudes are thought of not as existing independent of position and not as expressible by a unit, but only as regions in a manifold. Such inquiries have become a necessity for several parts of mathematics, namely for the treatment of many-valued analytic functions, and the lack of them is likely a principal reason why the celebrated theorem of Abel and the contributions of Lagrange, Pfaff, and Jacobi to the theory of differential equations have remained so long unfruitful. For the present purpose it will be sufficient to bring forward conspicuously two points out of this general part of the doctrine of extended magnitudes, wherein nothing further is assumed than what was already contained in the concept of it. The first of these will make plain how the notion of a multiply extended manifold came to exist; the second, the reference of the determination of place in a given manifold to determinations of quantity and the essential mark of an n-fold extension.

2

In a concept whose various modes of determination form a continuous manifold, if one passes in a definite way from one mode of determination to another, the modes of determination which are traversed constitute a simply extended manifold and its essential mark is this, that in it a continuous progress is possible from any point only in two directions, forward or backward. If now one forms the thought of this manifold again passing over into another entirely different, here again in a definite way, that is, in such a way that every point goes over into a definite point of the other, then will all the modes of determination thus obtained form a doubly extended manifold. In similar procedure one obtains a triply extended manifold when one represents to oneself that a double extension passes over in a definite way into one entirely different, and it is easy to see how one can prolong this construction indefinitely. If one considers his object of thought as variable instead of regarding the concept as determinable, then this construction can be characterized as a synthesis of a variability of n+1 dimensions out of a variability of n dimensions and a variability of one dimension.

3

I shall now show how one can conversely split up a variability, whose domain is given, into a variability of one dimension and a variability of fewer dimensions. To this end let one think of a variable portion of a manifold of one dimension,—reckoning from a fixed starting-point or origin, so that its values are comparable one with another—which has for every point of the given manifold a definite value changing continuously with that point; or in other words, let one assume within the given manifold a continuous function of place, and indeed a function such that it is not constant along any portion of this manifold. Every system of points in which the function has a constant value constitutes now a continuous manifold of fewer dimensions than that which was given. By change in the value of the function these manifolds pass over, one into another, continuously; hence one may assume that from one of them all the rest emanate, and this will come about, speaking generally, in such a way that every point of one passes over into a definite point of the other. Exceptional cases, and it is important to investigate them,—can be left out of consideration here. By this means the fixing of position in the given manifold is referred to the determination of one quantity and the fixing of position in a manifold of fewer dimensions. It is easy now to show that this latter has n-1 dimensions if the given manifold was n-fold extended. Hence by repetition of this procedure, to n times, the fixing of position in an n-dimensional manifold is reduced to n determinations of quantities, and therefore the fixing of position in a given manifold is reduced, whenever this is possible, to the determination of a finite number of quantities. There are however manifolds in which the fixing of position requires not a finite number but either an infinite series or a continuous manifold of determinations of quantity. Such manifolds are constituted for example by the possible determinations of a function for a given domain, the possible shapes of a figure in space, et cetera.

II. Relations of Measure, of Which an n-dimensional Manifold is Susceptible, on the Assumption that Lines Possess a Length Independent of Their Position; that is, that Every Line Can Be Measured by Every Other

Now that the concept of an n-fold extended manifold has been constructed and its essential mark has been found to be this, that the determination of position therein can be referred to n determinations of magnitude, there follows as second of the problems proposed above, an investigation into the relations of measure that such a manifold is susceptible of, also into the conditions which suffice for determining these metric relations. These relations of measure can be investigated only in abstract notions of magnitude and can be exhibited connectedly only in formulae; upon certain assumptions, however, one is able to resolve them into relations which are separately capable of being represented geometrically, and by this means it becomes possible to express geometrically the results of the calculation. Therefore if one is to reach solid ground, an abstract investigation in formulae is indeed unavoidable, but its results will allow an exhibition in the clothing of geometry. For both parts the foundations are contained in the celebrated treatise of Privy Councilor Gauss upon curved surfaces.

1

Determinations of measure require magnitude to be independent of location, a state of things which can occur in more than one way. The assumption that first offers itself, which I intend here to follow out, is perhaps this, that the length of lines be independent of their situation, that therefore every line be measurable by every other. If the fixing of the location is referred to determinations of magnitudes, that is, if the location of a point in the *n*-dimensional manifold be expressed by n variable quantities x_1, x_2, x_3, and so on to x_n, then the determination of a line will reduce to this, that the quantities x be given as functions of a single variable. The problem is then, to set up a mathematical expression for the length of lines, and for this purpose the quantities x must be thought of as expressible in units. This problem I shall treat only under certain restrictions, and limit myself first to such lines as have the ratios of the quantities dx—the corresponding changes in the quantities x—changing continuously; one can in that case think of the lines as laid off into elements within which the ratios of the quantities dx may be regarded as constant, and the problem reduces then to this: to set up for every point a general expression for a line-element which begins there, an expression which will therefore contain the quantities x and the quantities dx. In the second place I now assume that the length of the line-element, neglecting quantities of the second order, remains unchanged when all its points undergo infinitely small changes of position; in this it is implied that if all the quantities dx increase in the same ratio, the line-element likewise changes in this ratio. Upon these assumptions it will be possible for the line-element to be an arbitrary homogeneous function of the first degree in the quantities dx which remains unchanged when all the dx change sign, and in which the arbitrary constants are continuous functions of the quantities x. To find the simplest cases, I look first for an expression for the (n-l)-fold extended manifolds which are everywhere equally distant from the initial point of the line-element, that is, I look for a continuous function of place, which renders them distinct from one another. This will have to diminish or increase from the initial point out in all directions; I shall assume that it increases in all directions and therefore has a minimum in that point. If then its first and second differential quotients are finite, the differential of the first order must vanish and that of the second order must never be negative; I assume that it is always positive. This differential expression of the second order accordingly remains constant if ds remains constant, and increases in squared ratio when the quantities dx and hence also ds all change in the same ratio. That expression is therefore = const. ds^2, and consequently ds = the square root of an everywhere positive entire homogeneous function of the second degree in quantities dx having as coefficients continuous functions of the quantities x. For space this is, when one expresses the position of a point by rectangular coordinates, $ds = \sqrt{\Sigma(dx)^2}$ space is therefore comprised under this simplest case. The next case in order of simplicity would probably contain the manifolds in which the line-element can be expressed by the fourth root of a differential expression of the fourth degree. Investigation of this more general class indeed would require no essentially different principles, but would consume considerable time and throw relatively little new light upon the theory of space, particularly since the results cannot be expressed geometrically. I limit myself therefore to those manifolds in which the line-element is expressed by the square root of a differential expression of the second degree. Such an expression one can transform into another similar one by substituting for the n independent variables functions of n new independent variables. By this means however one cannot transform every expression into every other; for the expression contains (n)[(n+1)/2] coefficients which are arbitrary functions of the independent variables; but by introducing new variables one can satisfy only n relations (conditions), and so can make only n of the coefficients equal to given quantities. There remain then (n)[(n-1)/2] others completely determined by the nature of the manifold that is to be represented, and therefore for determining its metric relations (n)[(n-1)/2] functions of position are requisite. The manifolds in which, as in the plane and in space, the line-element can be reduced to the form $\sqrt{\Sigma(dx)^2}$ constitute therefore only a particular case of the manifolds under consideration here. They deserve a particular name, and I will therefore term *flat* these manifolds in which the square of the line-element can be reduced to the sum of squares of total differentials. Now in order to obtain a conspectus of the essential differences of the manifolds representable in this prescribed form it is necessary to remove those that spring from the mode of representation, and this is accomplished by choosing the variable quantities according to a definite principle.

2

For this purpose suppose the system of shortest lines emanating from an arbitrary point to have been constructed. The position of an undetermined point will then be determinable by specifying the direction of that shortest line in which it lies and its distance, in that line, from the starting-point; and it can therefore be expressed by the ratios of the quantities dx^0, that is the limiting ratios of the dx at the starting point of this shortest line and by the length s of this line. Introduce now instead of the dx^0 such linear expressions $d\alpha$ formed from them, that the initial value of the square of the line-element equals the sum of the squares of these expressions, so that the independent variables are: the quantity s and the ratios of quantities $d\alpha$. Finally, set in place of the $d\alpha$ such quantities proportional to them, x_1, x_2,… x_n, that the sum of their squares = s^2. After introducing these quantities, the square of the line-element for indefinitely small values of x becomes = $\Sigma(dx)^2$, and the term of next order in that $(ds)^2$ will be equal to a homogeneous expression of the second degree in the (n)[(n-1)/2] quantities (x_1dx_2-x_2dx_1), {x_1dx_3-x_3dx_1),..., that is, an indefinitely small quantity of dimension four; so that one obtains a finite magnitude when one divides it by the square of the indefinitely small triangle-area in whose vertices the values of the variables are (0, 0, 0,...), (x_1, x_2, x_3…), (dx_1, dx_2,.dx_3,...). This quantity retains the same value, so long as the quantities x and dx are contained in the same binary linear forms, or so long as the two shortest lines from the values 0 to the values x and from the values 0 to the values dx stay in the same element of surface, and it depends therefore only upon the place and the direction of that element. Plainly it is = 0 if the manifold represented is flat, that is if the square of the line-element is reducible to $\Sigma(dx)^2$, and it can accordingly be regarded as the measure of the divergence of the manifold from flatness in this point and in this direction of surface. Multiplied by -3/4 it becomes equal to the quantity which Privy Councilor Gauss has named the measure of curvature of a surface.

For determining the metric relations of an n-fold extended manifold representable in the prescribed form, in the foregoing discuss (n)[(n-1)/2] functions of position were found needful; hence when the measure of curvature in every point in (n)[(n-1)/2] surface-directions is given, from them can be determined the metric relations of the manifold, provided no identical relations exist among these values, and indeed in general this does not occur. The metric relations of these manifolds that have the line-element represented by the square root of a differential expression of the second degree can thus be expressed in a manner entirely independent of the choice of the variable quantities. A quite similar path to this goal can be laid out also in case of the manifolds in which the line-element is given in a less simple expression; *e. g.*, as the fourth root of a differential expression of the fourth degree. In that case the line-element, speaking generally, would no longer be reducible to the form of a square root of a sum of squares of differential expressions; and therefore in the expression for the square of the line-element the divergence from flatness would be an indefinitely small quantity of the dimension two, while in the former manifolds it was indefinitely small of the dimension four. This peculiarity of the latter manifolds may therefore well be called flatness in smallest parts. The most important peculiarity of these manifolds, for present purposes, on whose account solely they have been investigated here, is however this, that the relations of those doubly extended can be represented geometrically by surfaces, and those of more dimensions can be referred to those of the surfaces contained in them; and this requires still a brief elucidation.

3

In the conception of surfaces, along with the interior metric relations, in which only the length of the paths lying in them comes into consideration, there is always mixed also their situation with respect to points lying outside them. One can abstract however from external relations by carrying out such changes in the surfaces as leave unchanged the length of lines in them; *i. e.*, by thinking of them as bent in any arbitrary fashion,—without stretching—and by regarding all surfaces arising in this way one out of another as equivalent. , For example, arbitrary cylindrical or conical surfaces are counted as equivalent to a plane, because they can be formed out of it by mere bending, while interior metric relations remain unchanged; and all theorems regarding them—the whole of planimetry—retain their validity; on the other hand they count as essentially distinct from the sphere, which cannot be converted into a plane without stretching. According to the above investigation in every point the interior metric relations of a doubly extended manifold are characterized by the measure of curvature if the line-element can be expressed by the square root of a differential expression of the second degree, as is the case with surfaces. An intuitional significance can be given to this quantity in the case of surfaces, namely that it is the product of the two curvatures of the surface in this point; or also, that its product into an indefinitely small triangle-area formed of shortest lines is equal to half the excess of its angle-sum above two right angles, when measured in radians. The first definition would presuppose the theorem that the product of the two radii of curvature is not changed by merely bending a surface; the second, the theorem that at one and the same point the excess of the angle-sum of an indefinitely small triangle above two right angles is proportional to its area. To give a tangible meaning to measure of curvature of an *n*-dimensional manifold at a given point and in a surface direction passing through that point, it is necessary to start out from the principle that a shortest line, originating in a point, is fully determined when Its initial direction is given. According to this, a determinate surface is obtained when one prolongs into shortest lines all the initial directions going out from a point and lying in the given surface element; and this surface has in the given point a determinate measure of curvature, which is also the measure of curvature of the *n*-dimensional manifold in the given point and the given direction of surface.

4

Now before applications to space some considerations are needful regarding flat manifolds in general, *i. e.*, regarding those in which the square of the line-element is representable by a sum of squares of total differentials.

In a flat *n*-dimensional manifold the measure of curvature at every point is in every direction zero; but by the preceding investigation it suffices for determining the metric relations to know that at every point, in $(n)[(n-1)/2]$ surface directions whose measures of curvature are independent of one another, that measure is zero. Manifolds whose measure of curvature is everywhere zero may be regarded as a particular case of those manifolds whose curvature is everywhere constant. The common character of those manifolds of constant curvature can also be expressed thus: that the figures lying in them can be moved without stretching. For it is evident that the figures in them could not be pushed along and rotated at pleasure unless in every point the measure of curvature were the same in all directions. Upon the other hand, the metric relations of the manifold are completely determined by the measure of curvature. About any point, therefore, the metric relations in all directions are exactly the same as about any other point, and so the same constructions can be carried out from it, and consequently in manifolds with constant curvature every arbitrary position can be given to the figures. The metric relations of these manifolds depend only upon the value of the measure of curvature, and it may be mentioned, with reference to analytical presentation, that if one denotes this value by α, the expression for the line element can be given the form

$$1/(1+[(\alpha/4)\sqrt{\Sigma dx^2}])$$

5

Consideration of surfaces with constant measure of curvature can help toward a geometric exposition. It is easy to see that those surfaces whose curvature is positive will always permit themselves to be fitted upon a sphere whose radius is unity divided by the square root of the measure of curvature; but to visualize the complete manifold of these surfaces one should give to one of them the form of a sphere and to the rest the form of surfaces of rotation which touch it along the equator. Such surfaces as have greater curvature than this sphere will then touch the sphere from the inner side and take on a form like that exterior part of the surface of a ring which is turned away from the axis (remote from the axis); they could be shaped upon zones of spheres having a smaller radius, but would reach more than once around. Surfaces with lesser positive measure of curvature will be obtained by cutting out of spherical surfaces of greater radius a portion bounded by two halves of great circles, and making its edges adhere together. The surface with zero curvature will be simply a cylindrical surface standing upon the equator; the surfaces with negative curvature will be tangent to this cylinder externally and will be formed like the inner part of the surface of a ring, the part turned toward the axis.

If one thinks of these surfaces as loci for fragments of surface movable in them, as space is for bodies, then the fragments are movable in all these surfaces without stretching. Surfaces with positive curvature can always be formed in such wise that those fragments can be moved about without even bending, namely as spherical surfaces, not so however those with negative curvature. Beside this independence of position shown by fragments of surface, it is found in the surface with zero curvature that direction is independent of position, as is not true in the rest of the surfaces.

III. Application to Space

1

Following these investigations concerning the mode of fixing metric relations in an *n*-fold extended magnitude, the conditions can now be stated which are sufficient and necessary for determining metric relations in space, when it is assumed in advance that lines are independent of position and that the linear element is representable by the square root of a differential expression of the second degree; that is if flatness in smallest parts is assumed.

These conditions in the first place can be expressed thus: that the measure of the curvature in every point is equal to zero in three directions of surface; and therefore the metric relations of the space are determined when the sum of the angles in a triangle is everywhere equal to two right angles.

In the second place if one assumes at the start, like Euclid, an existence independent of situation not only for lines but also for bodies, then it follows that the measure of curvature is everywhere constant; and then the sum of the angles in all triangles is determined as soon as it is fixed for one triangle.

In the third place, finally, instead of assuming the length of lines to be independent of place and direction, one might even assume their length and direction to be independent of place. Upon this understanding the changes in place or differences in position are complex quantities expressible in three independent units.

2

In the course of preceding discussions, in the first place relations of extension (or of domain) were distinguished from those of measurement, and it was found that different relations of measure were conceivable along with identical relations of extension. Then were sought systems of simple determinations of measure by means of which the metric relations of space are completely determined and of which all theorems about such relations are a necessary consequence. It remains now to examine the question how, in what degree and to what extent these assumptions are guaranteed by experience. In this connection there subsists an essential difference between mere relations of extension and those of measurement: in the former, where the possible cases form a discrete manifold the declarations of experience are indeed never quite sure, but they are not lacking in exactness; while in the latter, where possible cases form a continuum, every determination based on experience remains always inexact, be the probability that it is nearly correct ever so great. This antithesis becomes important when these empirical determinations are extended beyond the limits of observation into the immeasurably great and the immeasurably small; for the second kind of relations obviously might become ever more inexact, beyond the bounds of observation, but not so the first kind.

When constructions in space are extended into the immeasurably great, unlimitedness must be distinguished from infiniteness; the one belongs to relations of extension, the other to those of measure. That space is an unlimited, triply extended manifold is an assumption applied in every conception of the external world; by it at every moment the domain of real perceptions is supplemented and the possible locations of an object that is sought for are constructed, and in these applications the assumption is continually being verified. The unlimitedness of space has therefore a greater certainty, empirically, than any experience of the external.

From this, however, follows in no wise its infiniteness, but on the contrary space would necessarily be finite, if one assumes that bodies are independent of situation and so ascribes to space a constant measure of curvature, provided this measure of curvature had any positive value however small. If one were to prolong the elements of direction, that lie in any element of surface, into shortest lines (geodetics), one would obtain an unlimited surface with constant positive measure of curvature, consequently a surface which would take on, in a triply extended manifold, the form of a spherical surface, and would therefore be finite.

3

Questions concerning the immeasurably large, are, for the explanation of Nature, useless questions. Quite otherwise is it however with questions concerning the immeasurably small. Knowledge of the causal connection of phenomena is based essentially upon the precision with which we follow them down into the infinitely small. The progress of recent centuries in knowledge of the mechanism of Nature has come about almost solely by the exactness of the syntheses rendered possible by the invention of Analysis of the infinite and by the simple fundamental concepts devised by Archimedes, Galileo, and Newton, and effectively employed by modern Physics. In the natural sciences however, where simple fundamental concepts are still lacking for such syntheses, one pursues phenomena into the spatially small, in order to perceive causal connections, just as far as the microscope permits. Questions concerning spatial relations of measure in the indefinitely small are therefore not useless.

If one premise that bodies exist independently of position, then the measure of curvature is everywhere constant; then from astronomical measurements it follows that it cannot differ from zero; at any rate its reciprocal value would have to be a surface in comparison with which the region accessible to our telescopes would vanish. If however bodies have no such non-dependence upon position, then one cannot conclude to relations of measure in the indefinitely small from those in the large. In that case the curvature can have at every point arbitrary values in three directions, provided only the total curvature of every metric portion of space be not appreciably different from zero. Even greater complications may arise in case the line element is not representable, as has been premised, by the square root of a differential expression of the second degree. Now however the empirical notions on which spatial measurements are based appear to lose their validity when applied to the indefinitely small, namely the concept of a fixed body and that of a light-ray; accordingly it is entirely conceivable that in the indefinitely small the spatial relations of size are not in accord with the postulates of geometry, and one would indeed be forced to this assumption as soon as it would permit a simpler explanation of the phenomena.

The question of the validity of the postulates of geometry in the indefinitely small is involved in the question concerning the ultimate basis of relations of size in space. In connection with this question, which may well be assigned to the philosophy of space, the above remark is applicable, namely that while in a discrete manifold the principle of metric relations is implicit in the notion of this manifold, it must come from somewhere else in the case of a continuous manifold. Either then the actual things forming the groundwork of a space must constitute a discrete manifold, or else the basis of metric relations must be sought for outside that actuality, in colligating forces that operate upon it.

A decision upon these questions can be found only by starting from the structure of phenomena that has been approved in experience hitherto, for which Newton laid the foundation, and by modifying this structure gradually under the compulsion of facts which it cannot explain. Such investigations as start out, like this present one, from general notions, can promote only the purpose that this task shall not be hindered by too restricted conceptions, and that progress in perceiving the connection of things shall not be obstructed by the prejudices of tradition.

This path leads out into the domain of another science, into the realm of physics, into which the nature of this present occasion forbids us to penetrate.